

Large- N limit of the gradient flow in the 2D $O(N)$ nonlinear sigma model

Hiroki Makino¹, Fumihiko Sugino², and Hiroshi Suzuki^{1,*}

¹*Department of Physics, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8581, Japan*

²*Okayama Institute for Quantum Physics, Kyoyama 1-9-1, Kita-ku, Okayama 700-0015, Japan*

*E-mail: hsuzuki@phys.kyushu-u.ac.jp

Received January 8, 2015; Revised February 19, 2015; Accepted March 6, 2015; Published April 15, 2015

.....
The gradient flow equation in the 2D $O(N)$ nonlinear sigma model with lattice regularization is solved in the leading order of the $1/N$ expansion. By using this solution, we analytically compute the thermal expectation value of a lattice energy–momentum tensor defined through the gradient flow. The expectation value reproduces thermodynamic quantities obtained by the standard large- N method. This analysis confirms that the above lattice energy–momentum tensor restores the correct normalization automatically in the continuum limit, in a system with a non-perturbative mass gap.
.....

Subject Index B31, B32, B34, B38

1. Introduction

The Yang–Mills gradient flow or the Wilson flow [1] is a powerful method to construct renormalized composite operators in gauge theory (see Ref. [2] for a recent review). This follows from the fact that a local product of bare fields evolved by the gradient flow possesses quite simple renormalization properties [3,4]: The multiplicative renormalization factor of the local product is determined simply by the number of fermion (or generally matter) fields contained in the local product; the flowed gauge field requires no multiplicative renormalization. Furthermore, no infinite subtraction is needed. Since such a renormalized operator is independent of regularization (after the parameter renormalization), the gradient flow is expected to be quite useful in relating physical quantities in continuum field theory and operators in lattice theory.

On the basis of this very general idea, a possible method to construct the energy–momentum tensor on the lattice through the gradient flow was proposed in Ref. [5]. This method was further investigated from a somewhat different perspective in Ref. [6] and also generalized in Ref. [7]. As well recognized [8,9], the construction of the energy–momentum tensor on the lattice is quite involved because lattice regularization breaks the translational invariance. The intention of Refs. [5,7] is that the constructed lattice energy–momentum tensor restores the correct normalization and the conservation law automatically in the continuum limit.

The construction in Refs. [5,7] is based on very natural assumptions, such as the existence of the energy–momentum tensor and the renormalizability of the gradient flow in the non-perturbative level. Also, the validity of the construction has been tested for thermodynamic quantities in quenched QCD by using a Monte Carlo simulation [10]. See also Ref. [11] for updated numerical results. However,

whether the conservation law is really restored in the non-perturbative level is still to be carefully examined.

Under these situations, it must be instructive to consider a simpler system that would allow a similar construction of the lattice energy–momentum tensor. Mainly with this motivation, the gradient flow for the 2D $O(N)$ nonlinear sigma model was investigated in Ref. [12]; an identical flow equation has also been studied in Ref. [13]. In Ref. [12], it was proven to all orders of perturbation theory that the N -vector field evolved by the gradient flow requires no multiplicative renormalization, a quite analogous property to the 4D gauge field. Because of this renormalizability of the gradient flow and because of the asymptotic freedom, one can imitate the construction of the lattice energy–momentum tensor in Refs. [5, 7]. Then, since the 2D $O(N)$ nonlinear sigma model is solvable in the $1/N$ expansion (see, e.g., Ref. [14]), one naturally expects that the property of the lattice energy–momentum tensor constructed through the gradient flow can be investigated by utilizing this analytical method, without any systematic errors associated with numerical study.

This is the main intention of the present paper: We test the construction of the lattice energy–momentum tensor in Ref. [12] by using the $1/N$ expansion. For this, we first recapitulate the well known large- N solution of the 2D $O(N)$ nonlinear sigma model that exhibits a non-perturbative mass gap (Sect. 2). Next, we solve the gradient flow equation in the leading order of the $1/N$ expansion (Sect. 3). We could not find a solution in the sub-leading order of the $1/N$ expansion. This is unfortunate, because in the leading order of the $1/N$ expansion all correlation functions factorize into one-point functions, while the test of the conservation law of the energy–momentum tensor requires nontrivial multi-point functions. Still, we can exactly compute one-point functions in the large- N limit. For example, we can obtain a non-perturbative running coupling constant by computing the vacuum expectation value of a composite operator analogous to the “energy density” defined in Ref. [1] (Sect. 4). The one-point function of our energy–momentum tensor is trivial in vacuum, but it becomes nontrivial if one considers the system at finite temperature, as in Ref. [10]. In Sect. 5, we compute the expectation value of the energy–momentum tensor at finite temperature in the large- N limit. This expectation value is directly related to thermodynamic quantities (the energy density and the pressure) of the present system. We observe that the expectation value correctly reproduces thermodynamic quantities directly computed by a standard statistical large- N method given in Appendix A. In Appendix B, we illustrate how a “naive” construction of the energy–momentum tensor on the lattice fails to reproduce the correct answer. The present analytical test confirms that the lattice energy–momentum tensor in Ref. [12] restores the correct normalization in this system with a non-perturbative mass gap, at least in the large- N limit. The last section is devoted to the conclusion.

2. Leading large- N solution of the 2D $O(N)$ nonlinear sigma model

The partition function of the 2D $O(N)$ nonlinear sigma model is given by

$$\begin{aligned} \mathcal{Z} = & \int \left[\prod_x d\sigma(x) \right] \left[\prod_x \prod_{i=1}^N dn^i(x) \right] \\ & \times \exp \left(-\frac{1}{2\lambda_0} a^2 \sum_x \left\{ \partial_\mu n^i(x) \partial_\mu n^i(x) + \sigma(x) \left[n^i(x) n^i(x) - N \right] \right\} \right), \end{aligned} \quad (2.1)$$

where λ_0 is the bare 't Hooft coupling constant, which is held fixed in the large- N limit. Throughout this paper, repeated Latin indices i, j, \dots , are assumed to be summed over the integers from 1 to N . In Eq. (2.1), we assume lattice regularization with the lattice spacing a and ∂_μ denotes the forward

difference operator. To apply the $1/N$ expansion (see, e.g., Ref. [14]), one first integrates over the N -vector field $n^i(x)$, to yield

$$\mathcal{Z} = \int \left[\prod_x d\sigma(x) \right] \exp \left\{ \frac{N}{2\lambda_0} a^2 \sum_x \sigma(x) - \frac{N}{2} \ln \det [-\partial_\mu^* \partial_\mu + \sigma(x)] \right\}, \quad (2.2)$$

where ∂_μ^* denotes the backward difference operator. Then, since the exponent is proportional to N , for large N , the integral over the auxiliary field $\sigma(x)$ can be evaluated by the saddle point method. Assuming that the saddle point is independent of x , $\sigma(x) = \sigma$, it is given by the gap equation,

$$\frac{1}{\lambda_0} = \int_p \frac{1}{\hat{p}^2 + \sigma}, \quad \int_p \equiv \int_{-\pi/a}^{\pi/a} \frac{d^2 p}{(2\pi)^2}, \quad (2.3)$$

where

$$\hat{p}^2 \equiv \sum_\mu \hat{p}_\mu \hat{p}_\mu, \quad \hat{p}_\mu \equiv \frac{2}{a} \sin \left(\frac{1}{2} a p_\mu \right). \quad (2.4)$$

An explicit momentum integration yields

$$\frac{1}{\lambda_0} = \int_p \frac{1}{\hat{p}^2 + \sigma} \xrightarrow{a \rightarrow 0} \frac{1}{4\pi} \left[-\ln(a^2 \sigma) + 5 \ln 2 \right]. \quad (2.5)$$

In the present problem, we may equally adopt dimensional regularization (DR), by setting the spacetime dimension $D = 2 - \epsilon$. With this regularization, the associated bare coupling constant λ_0^{DR} is renormalized as

$$\lambda_0^{\text{DR}} = \mu^\epsilon \lambda Z, \quad (2.6)$$

with the renormalization scale μ . The gap equation is obtained as Eq. (2.3) and one has

$$\frac{1}{\lambda_0^{\text{DR}}} = \frac{1}{\mu^\epsilon \lambda Z} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 + \sigma} \xrightarrow{D \rightarrow 2} \frac{1}{2\pi} \left[\frac{1}{\epsilon} - \frac{1}{2} \ln \left(\frac{e^\gamma \sigma}{4\pi} \right) \right], \quad (2.7)$$

where γ is the Euler constant. From this expression, we can deduce the exact renormalization constant in the minimal subtraction (MS) scheme,

$$Z^{-1} = 1 + \frac{\lambda}{2\pi} \frac{1}{\epsilon}, \quad (2.8)$$

and correspondingly the exact beta function,

$$\beta \equiv \mu \frac{\partial}{\partial \mu} \lambda \Big|_{\lambda_0^{\text{DR}} \text{ fixed}} = -\epsilon \lambda - \frac{\lambda^2}{2\pi}. \quad (2.9)$$

Then, from Eq. (2.7), we have

$$\sigma = 4\pi e^{-\gamma} \mu^2 e^{-4\pi/\lambda} = 4\pi e^{-\gamma} \Lambda^2, \quad \Lambda \equiv \mu e^{-2\pi/\lambda}, \quad (2.10)$$

in terms of the renormalized 't Hooft coupling λ in the MS scheme. Here, we have introduced the renormalization-group invariant scale parameter Λ in the MS scheme. Going back to Eq. (2.1), the saddle point value σ provides the mass gap for the N -vector field. This mass gap is non-perturbative, as the dependence of σ on the coupling constant λ shows.

3. Leading large- N solution of the gradient flow equation

Following Refs. [12,13], we consider the flow equation in the $O(N)$ nonlinear sigma model defined by¹

$$\partial_t n^i(t, x) = \partial_\mu^* \partial_\mu n^i(t, x) - \frac{1}{N} n^j(t, x) \partial_\mu^* \partial_\mu n^j(t, x) n^i(t, x), \quad (3.1)$$

where t is the flow time and the initial value at $t = 0$ is given by the N -vector field in the original $O(N)$ nonlinear sigma model,

$$n^i(t = 0, x) = n^i(x), \quad (3.2)$$

that is subject to the functional integral (2.1). In this expression, again, we are assuming lattice regularization in the x directions. To make the counting of the order of $1/N$ easier, we render the flow equation (3.1) linear in $n^i(t, x)$ by introducing a new variable $\sigma(t, x)$ as

$$\partial_t n^i(t, x) = \partial_\mu^* \partial_\mu n^i(t, x) - \sigma(t, x) n^i(t, x), \quad (3.3)$$

$$\sigma(t, x) = \frac{1}{N} n^j(t, x) \partial_\mu^* \partial_\mu n^j(t, x). \quad (3.4)$$

Note that the second relation does not contain the flow-time derivative. Then Eq. (3.3) can be formally solved as

$$n^i(t, x) = a^2 \sum_y \left[K_t(x - y) n^i(y) - \int_0^t ds K_{t-s}(x - y) \sigma(s, y) n^i(s, y) \right], \quad (3.5)$$

where

$$K_t(x) \equiv \int_p e^{ipx} e^{-t\hat{p}^2} \quad (3.6)$$

is the heat kernel with lattice regularization. The heat kernel satisfies $\partial_t K_t(x) = \partial_\mu^* \partial_\mu K_t(x)$ and $K_0(x) = \delta_{x,0}/a^2$. By iteratively solving Eq. (3.5), we can express the flowed field $n^i(t, x)$ in terms of the initial value $n^i(y)$ and $\sigma(s, z)$ at intermediate flow times as

$$\begin{aligned} n^i(t, x) = & \sum_{m=0}^{\infty} (-1)^m a^2 \sum_y a^2 \sum_{z_1} a^2 \sum_{z_2} \cdots a^2 \sum_{z_m} \\ & \times \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{m-1}} ds_m \sigma(s_1, z_1) \sigma(s_2, z_2) \cdots \sigma(s_m, z_m) \\ & \times K_{t-s_1}(x - z_1) K_{s_1-s_2}(z_1 - z_2) \cdots K_{s_{m-1}-s_m}(z_{m-1} - z_m) K_{s_m}(z_m - y) n^i(y). \end{aligned} \quad (3.7)$$

Diagrammatic representation of the above elements and expressions is useful.² In Eq. (3.7), the heat kernel $K_t(x)$ (3.6) connecting two spacetime points is represented by an arrowed solid line as Fig. 1. An open circle denotes the interaction between the flowed N -vector field and the auxiliary field $\sigma(t, x)$, which is represented by a short dotted line. A typical term in the solution (3.7) is thus represented as Fig. 2, where the N -vector field at the zero flow time, $n^i(y)$, is represented by the cross. The equality (3.4) is, on the other hand, represented as Fig. 3, where two short solid lines represent two N -vector fields in the right-hand side of Eq. (3.4). Note that Eq. (3.4) and thus the symbol in Fig. 3 carry the factor $1/N$.

We may now substitute the solution (3.7) in the equality (3.4) to express the auxiliary field $\sigma(t, x)$ in terms of the initial value $n^i(y)$. This process can be diagrammatically represented as Fig. 4.

¹ Note that the normalization of the N -vector field is different from that of Ref. [12] by the factor $1/\sqrt{N}$.

² The present convention for the “flow Feynman diagram” is quite different from that in Ref. [12].



Fig. 1. Diagrammatic representation of the heat kernel (3.6).

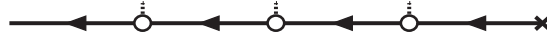


Fig. 2. The $m = 3$ term in the solution (3.7). The cross denotes the N -vector field at zero flow time, $n^i(y)$.



Fig. 3. Diagrammatic representation of the equality (3.4), which is $O(1/N)$.

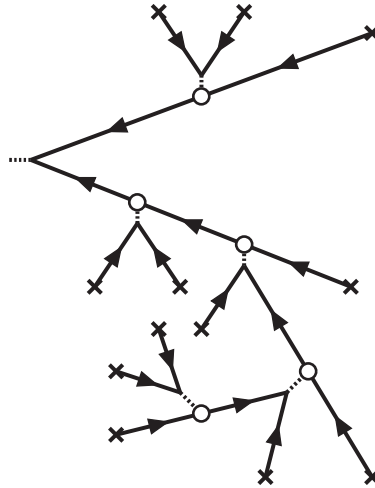


Fig. 4. $\sigma(t, x)$ in terms of the zero flow-time field $n^i(x)$.

So far, everything concerns the solution to the deterministic differential equation (3.1). Let us now take into account the quantum effect, i.e., the fact that the initial value $n^i(y)$ is subject to the quantum average (2.1). In the leading order of the $1/N$ expansion, the integration over the auxiliary field $\sigma(x)$ in Eq. (2.1) is approximated by the value at the saddle point, $\sigma(x) = \sigma$. Then, since the action is quadratic in $n^i(x)$, the quantum average produces contractions of $n^i(x)$ fields by the free massive propagator with the mass σ . In terms of the diagrammatic representation above, this amounts to taking the contraction of all crosses in all possible ways. Let us consider these contractions for $\sigma(t, x)$ in Fig. 4. In this diagram, recalling that the vertex in Fig. 3 carries the factor $1/N$ and noting that each closed loop of the N -vector field gains the factor N , it is obvious that the leading large- N contribution to the quantum average of $\sigma(t, x)$, denoted by $\langle \sigma(t, x) \rangle$, is given by a diagram such as Fig. 5 in which each closed loop contains only one vertex in Fig. 3; overall, this is a quantity of $O(N^0)$.³ The topology of diagrams in the leading order in the $1/N$ expansion is thus identical to that of the leading order diagrams in the conventional $1/N$ expansion of the N -vector model (the so-called “cactus” diagrams). To calculate sub-leading orders of $1/N$, we have to find not only

³ In the diagrammatic representation, we adopt a rule [3] that arrows are removed when end points of arrowed lines are contracted.

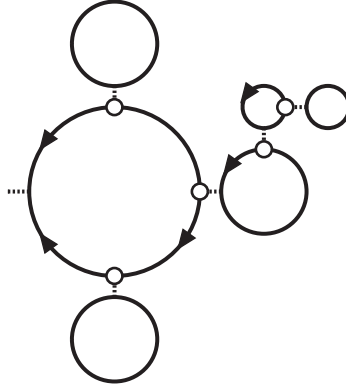


Fig. 5. $\langle \sigma(t, x) \rangle$ in the leading order of the $1/N$ expansion; this diagram is obtained by taking the contraction of the $n^i(y)$ in Fig. 4.

the one-point function but also the (connected) higher-point functions of $\sigma(t, x)$, whose systematic treatment is left as a future subject.

In a similar manner, it is easy to see that, in the leading order in the $1/N$ expansion, a correlation function of generic operators containing $\sigma(t, x)$ and $n^i(t, x)$ fields factorizes into the product of the expectation value $\langle \sigma(t, x) \rangle$ and correlation functions of the $n^i(t, x)$; this is nothing but the large- N factorization. Then, since $\langle \sigma(t, x) \rangle$ is independent of the spacetime position x (the external momentum in Fig. 5 is zero), we can set $\sigma(s, z)$ in Eq. (3.7) constant in spacetime, $\sigma(s, z) \rightarrow \langle \sigma(s) \rangle$.⁴ Then, noting the relation

$$a^2 \sum_z K_{t-u}(x-z) K_{u-s}(z-y) = K_{t-s}(x-y), \quad (3.8)$$

we have a compact expression for Eq. (3.7),

$$n^i(t, x) = e^{-\int_0^t ds \sigma(s)} a^2 \sum_y K_t(x-y) n^i(y), \quad (3.9)$$

where we have written $\sigma(s) \equiv \langle \sigma(s) \rangle$ for notational simplicity. The propagator between the flowed N -vector fields is then obtained by contracting $n^i(y)$ in Eq. (3.9) by the propagator in the large- N limit:

$$\langle n^i(x) n^j(y) \rangle = \delta^{ij} \lambda_0 \int_p e^{ip(x-y)} \frac{1}{\hat{p}^2 + \sigma}. \quad (3.10)$$

This yields

$$\langle n^i(t, x) n^j(s, y) \rangle = \delta^{ij} e^{-\int_0^t du \sigma(u)} e^{-\int_0^s dv \sigma(v)} \lambda_0 \int_p e^{ip(x-y)} \frac{e^{-(t+s)\hat{p}^2}}{\hat{p}^2 + \sigma}. \quad (3.11)$$

In terms of this “dressed propagator”, the expectation value $\langle \sigma(t) \rangle$ is given from Eq. (3.4) by

$$\begin{aligned} \sigma(t) &= \left\langle \frac{1}{N} n^i(t, x) \partial_\mu^* \partial_\mu n^i(t, x) \right\rangle \\ &= e^{-2 \int_0^t ds \sigma(s)} \lambda_0 \int_p \frac{-\hat{p}^2}{\hat{p}^2 + \sigma} e^{-2t\hat{p}^2}. \end{aligned} \quad (3.12)$$

This self-consistency condition is schematically represented as Fig. 6.

⁴ Note that, since there is no translational invariance in the flow-time direction (the zero flow time is a very special point), we cannot assume that $\langle \sigma(s) \rangle$ is independent of s . In fact, we will shortly see that $\langle \sigma(s) \rangle$ possesses nontrivial s dependence.

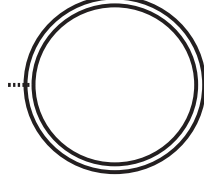


Fig. 6. Figure 5 in terms of the dressed propagator (3.11) (the doubled line).

Now we solve the self-consistency condition for $\sigma(t)$, Eq. (3.12). For this, we introduce

$$\Sigma(t) = \int_0^t ds \sigma(s), \quad (3.13)$$

and write Eq. (3.12) as

$$e^{2\Sigma(t)} \frac{d\Sigma(t)}{dt} = \lambda_0 \int_p \frac{-\hat{p}^2}{\hat{p}^2 + \sigma} e^{-2t\hat{p}^2}. \quad (3.14)$$

As far as lattice regularization is understood, the momentum integration in the right-hand side is regular even at $t = 0$ and we may integrate both sides of the above relation over t from $t = 0$ to some prescribed value. In this way, we have

$$\Sigma(t) = \frac{1}{2} \ln \left(\lambda_0 \int_p \frac{e^{-2t\hat{p}^2}}{\hat{p}^2 + \sigma} \right), \quad (3.15)$$

where we have used the saddle point condition (2.3). Substituting this back into Eq. (3.12) leads to

$$\sigma(t) = \sigma - \frac{\int_p e^{-2t\hat{p}^2}}{\int_p \frac{e^{-2t\hat{p}^2}}{\hat{p}^2 + \sigma}}. \quad (3.16)$$

As far as $t > 0$, the integrals are well convergent and we may send $a \rightarrow 0$ to have a definite continuum limit. Thus, for $t > 0$, we obtain

$$\sigma(t) \xrightarrow{a \rightarrow 0} \sigma - \frac{1}{2te^{2\sigma t} \Gamma(0, 2\sigma t)} \xrightarrow{t \rightarrow 0} \sigma + \frac{1}{2t \ln(2e^\gamma \sigma t)} [1 - 2\sigma t + O(t/\ln t)], \quad (3.17)$$

where $\Gamma(z, p)$ is the incomplete gamma function. Here, the order of the two limits is very important. Our construction of the energy-momentum tensor on the basis of the gradient flow relies on a universality, which is ensured if the flow time is fixed and ultraviolet regularization is removed. Thus, we should first take the continuum limit while keeping the flow time finite; we then consider the small flow-time limit. Also, using Eqs. (3.12) and (3.16), for $t > 0$ we have

$$e^{-2 \int_0^t ds \sigma(s)} \lambda_0 = \frac{1}{\int_p \frac{e^{-2t\hat{p}^2}}{\hat{p}^2 + \sigma}} \xrightarrow{a \rightarrow 0} \frac{4\pi}{e^{2\sigma t} \Gamma(0, 2\sigma t)} \xrightarrow{t \rightarrow 0} -\frac{4\pi}{\ln(2e^\gamma \sigma t)} [1 - 2\sigma t + O(t/\ln t)]. \quad (3.18)$$

The dressed propagator (3.11) with this prefactor provides the solution of the gradient flowed system at the leading order in the large- N limit.

4. Non-perturbative running coupling in the large- N limit

Since the expectation value,

$$\lambda_R(1/\sqrt{8t}) \equiv 16\pi t \langle E(t, x) \rangle, \quad (4.1)$$

where

$$E(t, x) \equiv \frac{1}{2} \frac{1}{N} \partial_\mu n^i(t, x) \partial_\mu n^i(t, x), \quad (4.2)$$

is a renormalized quantity [12] that possesses the perturbative expansion, $16\pi t \langle E(t, x) \rangle = \lambda_0 + \dots$, it can be used as a non-perturbative definition of the running coupling constant at the renormalization scale $1/\sqrt{8t}$ [12]. This is analogous to the non-perturbative running gauge coupling defined through the “energy density operator” [1].

From our large- N solution in the previous section, we have

$$\lambda_R(1/\sqrt{8t}) = -8\pi t \sigma(t) \xrightarrow{a \rightarrow 0} -8\pi \sigma t + \frac{4\pi}{e^{2\sigma t} \Gamma(0, 2\sigma t)}, \quad t > 0. \quad (4.3)$$

This is a monotonically increasing function of t being consistent with the fact that the exact beta function (2.9) is negative definite.

5. Thermal expectation value of the lattice energy–momentum tensor

Following the general idea in Refs. [5,7], a possible method using the gradient flow to construct a lattice energy–momentum tensor for the $O(N)$ nonlinear sigma model has been proposed [12]. The intention in Ref. [12] is to construct a lattice operator that restores the correct normalization and the conservation law automatically in the continuum limit. It is thus quite interesting to examine if the idea works (or not) by using the above exact large- N solution of the gradient flow. Unfortunately, at the leading order of the $1/N$ expansion, any correlation function factorizes into one-point functions of $O(N)$ invariant quantities. Thus, in the present paper, we can consider only the one-point function of the energy–momentum tensor. Since we define the energy–momentum tensor by subtracting the vacuum expectation value,

$$\{T_{\mu\nu}\}_R(x) \equiv T_{\mu\nu}(x) - \langle T_{\mu\nu}(x) \rangle, \quad (5.1)$$

the one-point function is trivial in the vacuum. The one-point function of the energy–momentum tensor is quite interesting, however, if we consider the system at finite temperature, as in Ref. [10]. Thus, let us consider the expectation value of the energy–momentum tensor at finite temperature. The construction in Ref. [12] adopted in the present large- N limit reads

$$\begin{aligned} \{T_{\mu\nu}\}_R(x) = \lim_{t \rightarrow 0} \lim_{a \rightarrow 0} & \left\{ c_1(t) \left[\partial_\mu n^i(t, x) \partial_\nu n^i(t, x) - \frac{1}{2} \delta_{\mu\nu} \partial_\rho n^i(t, x) \partial_\rho n^i(t, x) \right] \right. \\ & \left. + c_2(t) \left[\frac{1}{2} \delta_{\mu\nu} \partial_\rho n^i(t, x) \partial_\rho n^i(t, x) - \left\langle \frac{1}{2} \delta_{\mu\nu} \partial_\rho n^i(t, x) \partial_\rho n^i(t, x) \right\rangle \right] \right\}, \end{aligned} \quad (5.2)$$

where the coefficients are given by

$$c_1(t) = \frac{1}{\bar{\lambda}(1/\sqrt{8t})} - \frac{1}{4\pi} \ln \pi + O(\bar{\lambda}), \quad c_2(t) = \frac{1}{4\pi} - \frac{1}{(4\pi)^2} \bar{\lambda}(1/\sqrt{8t}) + O(\bar{\lambda}^2), \quad (5.3)$$

and

$$\bar{\lambda}(q) = -\frac{4\pi}{\ln(\Lambda^2/q^2)} \quad (5.4)$$

is the running coupling constant at the renormalization scale q . From the expressions in Ref. [12] (with the normalization change $n^i(t, x) \rightarrow n^i(t, x)/\sqrt{N}$), these expressions are obtained by setting $g^2 = \lambda/N$ and taking $N \rightarrow \infty$.

The expectation value of the energy–momentum tensor at finite temperature,

$$\langle \{T_{\mu\nu}\}_R(x) \rangle_\beta, \quad (5.5)$$

where β is the inverse temperature, is then obtained by contracting $n^i(t, x)$ by the dressed propagator (3.11) with the periodic boundary condition in the Euclidean time direction x_0 ; the time component of the momentum in Eq. (3.11) is thus quantized to the Matsubara frequency:

$$p_0 = \omega_n \equiv \frac{2\pi n}{\beta}, \quad n \in \mathbb{Z}. \quad (5.6)$$

Thus, for instance, we have

$$\begin{aligned} & \left\langle \partial_0 n^i(t, x) \partial_0 n^i(t, x) \right\rangle_\beta \\ &= N e^{-2 \int_0^t ds \sigma_\beta(s)} \lambda_0 \frac{1}{\beta} \sum_{-\pi/a < \omega_n < \pi/a} \int_{-\pi/a}^{\pi/a} \frac{dp_1}{2\pi} \frac{\hat{\omega}_n^2}{\hat{\omega}_n^2 + \hat{p}_1^2 + \sigma_\beta} e^{-2t(\hat{\omega}_n^2 + \hat{p}_1^2)}, \end{aligned} \quad (5.7)$$

where $\sigma_\beta(s)$ is the flow-time-dependent auxiliary field at finite temperature that fulfills a finite temperature counterpart of Eq. (3.18):

$$e^{-2 \int_0^t ds \sigma_\beta(s)} \lambda_0 = \frac{1}{\frac{1}{\beta} \sum_{-\pi/a < \omega_n < \pi/a} \int_{-\pi/a}^{\pi/a} \frac{dp_1}{2\pi} \frac{e^{-2t(\hat{\omega}_n^2 + \hat{p}_1^2)}}{\hat{\omega}_n^2 + \hat{p}_1^2 + \sigma_\beta}}. \quad (5.8)$$

On the other hand, σ_β is the saddle point value of the auxiliary field at finite temperature, which is given by

$$\frac{1}{\lambda_0} = \frac{1}{\beta} \sum_{-\pi/a < \omega_n < \pi/a} \int_{-\pi/a}^{\pi/a} \frac{dp_1}{2\pi} \frac{1}{\hat{\omega}_n^2 + \hat{p}_1^2 + \sigma_\beta}. \quad (5.9)$$

Now, in expressions such as Eqs. (5.7) and (5.8), the sum and the integral are well convergent for $t > 0$ because of the Gaussian damping factor. Thus we may simply remove lattice regularization in those expressions to yield regularization-independent expressions such as

$$\left\langle \partial_0 n^i(t, x) \partial_0 n^i(t, x) \right\rangle_\beta = N e^{-2 \int_0^t ds \sigma_\beta(s)} \lambda_0 \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \frac{\omega_n^2}{\omega_n^2 + p_1^2 + \sigma_\beta} e^{-2t(\omega_n^2 + p_1^2)}, \quad (5.10)$$

$$\left\langle \partial_1 n^i(t, x) \partial_1 n^i(t, x) \right\rangle_\beta = N e^{-2 \int_0^t ds \sigma_\beta(s)} \lambda_0 \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \frac{p_1^2}{\omega_n^2 + p_1^2 + \sigma_\beta} e^{-2t(\omega_n^2 + p_1^2)}, \quad (5.11)$$

and

$$e^{-2 \int_0^t ds \sigma_\beta(s)} \lambda_0 = \frac{1}{\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \frac{1}{\omega_n^2 + p_1^2 + \sigma_\beta} e^{-2t(\omega_n^2 + p_1^2)}}. \quad (5.12)$$

These clearly illustrate the “UV finiteness” of the gradient flow: Any correlation function of the flowed N -vector field in terms of the renormalized coupling is UV finite without the wave function

renormalization [12].⁵ It is the basic idea for the construction of the lattice energy–momentum tensor in Refs. [5,7,12] that the continuum limit $a \rightarrow 0$ of a lattice composite operator of the flowed field reduces to a regularization-independent expression. Thus, we have observed that the continuum limit $a \rightarrow 0$ in Eq. (5.2) can be almost trivially taken. Next, to consider the small flow-time limit $t \rightarrow 0$ in Eq. (5.2), we estimate the sum and the integral appearing in the above expressions for $t \rightarrow 0$. This can be accomplished by noting the Poisson resummation formula,

$$\sum_{n=-\infty}^{\infty} e^{-\alpha n^2} = \sqrt{\frac{\pi}{\alpha}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2 / \alpha}, \quad (5.13)$$

and, after some calculation, we have the following asymptotic expansions for $t \rightarrow 0$:

$$\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} e^{-2t(\omega_n^2 + p_1^2)} \sim \frac{1}{4\pi} \frac{1}{2t}, \quad (5.14)$$

$$\begin{aligned} & \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \frac{1}{\omega_n^2 + p_1^2 + \sigma_\beta} e^{-2t(\omega_n^2 + p_1^2)} \\ & \sim -\frac{1}{4\pi} \ln(2e^\gamma \sigma_\beta t) + \frac{1}{\pi} \sum_{n=1}^{\infty} K_0(\beta \sqrt{\sigma_\beta} n) \\ & \quad - \frac{1}{2\pi} \sigma_\beta t [\ln(2e^\gamma \sigma_\beta t) - 1] + \frac{2}{\pi} \sigma_\beta t \sum_{n=1}^{\infty} K_0(\beta \sqrt{\sigma_\beta} n) + O(t^2 \ln t), \end{aligned} \quad (5.15)$$

$$\begin{aligned} & \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \frac{\omega_n^2}{\omega_n^2 + p_1^2 + \sigma_\beta} e^{-2t(\omega_n^2 + p_1^2)} \\ & \sim \frac{1}{8\pi} \left[\frac{1}{2t} + \sigma_\beta \ln(2e^\gamma \sigma_\beta t) \right] + \frac{1}{\pi} \sigma_\beta \sum_{n=1}^{\infty} \left[\frac{1}{\beta \sqrt{\sigma_\beta} n} K_1(\beta \sqrt{\sigma_\beta} n) - K_2(\beta \sqrt{\sigma_\beta} n) \right] + O(t \ln t), \end{aligned} \quad (5.16)$$

$$\begin{aligned} & \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \frac{p_1^2}{\omega_n^2 + p_1^2 + \sigma_\beta} e^{-2t(\omega_n^2 + p_1^2)} \\ & \sim \frac{1}{8\pi} \left[\frac{1}{2t} + \sigma_\beta \ln(2e^\gamma \sigma_\beta t) \right] + \frac{1}{\pi} \sigma_\beta \sum_{n=1}^{\infty} \frac{1}{\beta \sqrt{\sigma_\beta} n} K_1(\beta \sqrt{\sigma_\beta} n) + O(t \ln t), \end{aligned} \quad (5.17)$$

where $K_n(z)$ denotes the modified Bessel function of the n th kind. At this stage, we note the following relation:

$$-\frac{1}{4\pi} \ln(2e^\gamma \sigma_\beta t) + \frac{1}{\pi} \sum_{n=1}^{\infty} K_0(\beta \sqrt{\sigma_\beta} n) = -\frac{1}{4\pi} \ln(2e^\gamma \sigma t), \quad (5.18)$$

⁵ From Eqs. (2.5) and (5.9), one sees that the ratio between σ_β and σ is a UV convergent quantity that is independent of the regularization; the explicit relation is given by (5.18). Thus, as long as we renormalize the bare coupling constants λ_0 and λ_0^{DR} so that σ in Eqs. (2.5) and (2.7) are identical, σ_β defined in Eq. (5.9) through lattice regularization and σ_β defined in Eq. (A2) through dimensional regularization are identical; σ_β is, of course, finite after the renormalization.

which can be obtained by comparing two gap equations, Eqs. (2.7) and (A7). By using this in Eq. (5.15) and then in Eq. (5.12), we find the asymptotic behavior of the prefactor for $t \rightarrow 0$:

$$e^{-2 \int_0^t ds \sigma_\beta(s)} \lambda_0 \sim -\frac{4\pi}{\ln(2e^\gamma \sigma t)} [1 - 2\sigma_\beta t + O(t/\ln t)]. \quad (5.19)$$

Also, from Eqs. (5.3), (5.4), and (2.10), for $t \rightarrow 0$,

$$c_1(t) \sim -\frac{1}{4\pi} \ln(2e^\gamma \sigma t) + O(1/\ln t), \quad c_2(t) \sim \frac{1}{4\pi} \left[1 + \frac{1}{\ln(2e^\gamma \sigma t/\pi)} \right] + O(1/\ln^2 t). \quad (5.20)$$

It is now straightforward to obtain the $t \rightarrow 0$ limit in Eq. (5.2). Noting that $\beta \rightarrow \infty$ and $\sigma_\beta \rightarrow \sigma$ on the vacuum, we have

$$\langle \{T_{00}\}_R(x) \rangle_\beta = -\frac{N}{8\pi}(\sigma_\beta - \sigma) - \frac{N}{2\pi} \sigma_\beta \sum_{n=1}^{\infty} K_2(\beta \sqrt{\sigma_\beta} n), \quad (5.21)$$

$$\langle \{T_{11}\}_R(x) \rangle_\beta = -\frac{N}{8\pi}(\sigma_\beta - \sigma) + \frac{N}{2\pi} \sigma_\beta \sum_{n=1}^{\infty} K_2(\beta \sqrt{\sigma_\beta} n), \quad (5.22)$$

$$\langle \{T_{01}\}_R(x) \rangle_\beta = 0. \quad (5.23)$$

In this calculation, one finds that $1/t$ singularities are canceled between the expectation value at finite temperature and the vacuum expectation value, and a finite small flow-time limit results.

The thermodynamic quantities, the energy density ε and the pressure P , are related to these expectation values of the energy–momentum tensor as

$$\varepsilon = -\langle \{T_{00}\}_R(x) \rangle_\beta \quad \text{and} \quad P = \langle \{T_{11}\}_R(x) \rangle_\beta. \quad (5.24)$$

In Appendix A, we compute these thermodynamic quantities by the standard large- N method. We find that Eq. (5.24) with Eqs. (5.21) and (5.22) correctly reproduces those large- N results.

6. Conclusion

In the present paper, we solved the gradient flow equation for the 2D $O(N)$ nonlinear sigma model in the leading order of the large- N expansion. By using this solution, one can non-perturbatively compute one-point functions of $O(N)$ invariant composite operators made from the flowed N -vector field in the large- N limit. We computed a non-perturbative running coupling from the expectation value of the “energy density operator” in which the flow time gives the renormalization scale. We also computed the thermal expectation value of the lattice energy–momentum tensor, which is defined by a small flow time limit of composite operators of the flowed field [12]. We found that the small flow time limit can be taken as expected and the lattice energy–momentum tensor correctly reproduces the thermodynamic quantities obtained by the standard large- N approximation. This result for the present system with a non-perturbatively generated mass gap strongly supports the correctness of the reasoning for the lattice energy–momentum tensor in Refs. [5, 7, 12].

Quite unfortunately, in the present work, we could not find the solution for the gradient flow equation in the next-to-leading order of the large- N expansion. If this solution is obtained, it will make possible the examination of the conservation law of the lattice energy–momentum tensor. We hope to come back to this problem in the near future.

Acknowledgements

We would like to thank Kengo Kikuchi for the discussion. F.S. would like to thank the members of KIAS, especially Hyeonjoon Shin, for their warm hospitality during his visit. The work of F.S. and H.S. is supported in part by Grants-in-Aid for Scientific Research 25400289 and 23540330, respectively.

Note added

In a recent paper [15], some of the results presented in this paper have been obtained independently.

Funding

Open Access funding: SCOAP³.

Appendix A. Thermodynamics at large N

In the large- N limit, the free energy density of the 2D $O(N)$ nonlinear sigma model at finite temperature is given by, as a natural generalization of the zero-temperature expression (2.2),

$$f(\beta) = -\frac{N}{2\lambda_0}\beta\sigma_\beta + \frac{N}{2} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \ln(\omega_n^2 + p_1^2 + \sigma_\beta), \quad \omega_n \equiv \frac{2\pi}{\beta}n, \quad (\text{A1})$$

where σ_β denotes the saddle point value of the auxiliary field $\sigma(x)$ at finite temperature, which is given by the solution of the finite temperature gap equation:

$$\frac{1}{\lambda_0} = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{dp_1}{2\pi} \frac{1}{\omega_n^2 + p_1^2 + \sigma_\beta}. \quad (\text{A2})$$

We regularize the formal expressions (A1) and (A2) by using dimensional regularization. For this, we note the identity

$$\frac{1}{\beta} \sum_{n=-\infty}^{\infty} F(\omega_n) = \sum_{n=-\infty}^{\infty} \int \frac{dp_0}{2\pi} e^{ip_0\beta n} F(p_0), \quad (\text{A3})$$

and regularize Eq. (A1) as

$$f(\beta) \equiv -\frac{N}{2\lambda_0^{\text{DR}}}\beta\sigma_\beta + \frac{N}{2}\beta \sum_{n=-\infty}^{\infty} \int \frac{d^D p}{(2\pi)^D} e^{ip_0\beta n} \ln(p^2 + \sigma_\beta), \quad (\text{A4})$$

where λ_0^{DR} is the bare coupling constant in dimensional regularization appearing in Eq. (2.7), and Eq. (A2) as

$$\frac{1}{\lambda_0^{\text{DR}}} = \sum_{n=-\infty}^{\infty} \int \frac{d^D p}{(2\pi)^D} e^{ip_0\beta n} \frac{1}{p^2 + \sigma_\beta}. \quad (\text{A5})$$

In the second term of the right-hand side of Eq. (A4), only the $n = 0$ term requires regularization because the $n \neq 0$ terms are Fourier transformations and UV convergent. After the momentum integration, we have

$$f(\beta) = -\frac{N}{2\lambda_0^{\text{DR}}}\beta\sigma_\beta + \frac{N}{4\pi}\beta\sigma_\beta \left\{ \frac{1}{\epsilon} - \frac{1}{2} \left[\ln\left(\frac{e^\gamma \sigma_\beta}{4\pi}\right) - 1 \right] \right\} - \frac{N}{\pi}\beta\sigma_\beta \sum_{n=1}^{\infty} \frac{K_1(\beta\sqrt{\sigma_\beta}n)}{\beta\sqrt{\sigma_\beta}n}. \quad (\text{A6})$$

Similarly, the integration in Eq. (A5) yields

$$\frac{1}{\lambda_0^{\text{DR}}} = \frac{1}{2\pi} \left[\frac{1}{\epsilon} - \frac{1}{2} \ln\left(\frac{e^\gamma \sigma_\beta}{4\pi}\right) \right] + \frac{1}{\pi} \sum_{n=1}^{\infty} K_0(\beta\sqrt{\sigma_\beta}n). \quad (\text{A7})$$

Plugging this into Eq. (A6), by noting the identity $K_0(z) - K_2(z) = -(2/z)K_1(z)$, we have

$$f(\beta) = \beta \left[\frac{N}{8\pi} (\sigma_\beta - \sigma) - \frac{N}{2\pi} \sigma_\beta \sum_{n=1}^{\infty} K_2(\beta\sqrt{\sigma_\beta}n) \right], \quad (\text{A8})$$

where we have shifted the origin of the free energy density by $-\beta(N/8\pi)\sigma$, so that it vanishes at zero temperature as $\lim_{\beta \rightarrow \infty} f(\beta)/\beta = 0$; note that $\lim_{\beta \rightarrow \infty} \sigma_\beta = \sigma$ and

$\lim_{\beta \rightarrow \infty} \sum_{n=1}^{\infty} K_2(\beta \sqrt{\sigma_\beta} n) = 0$. Since the pressure P is related to the free energy density as $P = -f(\beta)/\beta$ in the thermodynamic limit, we have

$$P = -\frac{N}{8\pi} (\sigma_\beta - \sigma) + \frac{N}{2\pi} \sigma_\beta \sum_{n=1}^{\infty} K_2(\beta \sqrt{\sigma_\beta} n). \quad (\text{A9})$$

On the other hand, the energy density is given from the free energy density by $\varepsilon = \partial f(\beta)/\partial \beta$. The derivative of Eq. (A8) with respect to β contains $\partial \sigma_\beta / \partial \beta$, which can be deduced from the β derivative of Eq. (A7) as

$$\beta \frac{\partial \sigma_\beta}{\partial \beta} = -\sigma_\beta \frac{4 \sum_{n=1}^{\infty} \beta \sqrt{\sigma_\beta} n K_1(\beta \sqrt{\sigma_\beta} n)}{1 + 2 \sum_{n=1}^{\infty} \beta \sqrt{\sigma_\beta} n K_1(\beta \sqrt{\sigma_\beta} n)}, \quad (\text{A10})$$

where we have used the relation $K'_0(z) = -K_1(z)$. Using this expression and noting the identity $z K'_2(z) + 2K_2(z) = -z K_1(z)$, we finally obtain

$$\varepsilon = \frac{N}{8\pi} (\sigma_\beta - \sigma) + \frac{N}{2\pi} \sigma_\beta \sum_{n=1}^{\infty} K_2(\beta \sqrt{\sigma_\beta} n). \quad (\text{A11})$$

Comparing Eqs. (A9) and (A11) with Eq. (5.24) given by Eqs. (5.21) and (5.22), we find that our lattice energy–momentum tensor in the continuum limit correctly reproduces those thermodynamic quantities.

Appendix B. Naive lattice energy–momentum tensor

It is interesting to see how the following “naive” energy–momentum tensor,⁶

$$T_{\mu\nu}^{\text{naive}}(x) = \frac{1}{\lambda_0} \left[\partial_\mu n^i(x) \partial_\nu n^i(x) - \frac{1}{2} \delta_{\mu\nu} \partial_\rho n^i(x) \partial_\rho n^i(x) \right], \quad (\text{B1})$$

when used in conjunction with lattice regularization, fails to reproduce the correct answer.

Using the propagator (3.10), the thermal expectation value of Eq. (B1) is given by

$$\left\langle T_{00}^{\text{naive}}(x) \right\rangle_\beta = -\left\langle T_{11}^{\text{naive}}(x) \right\rangle_\beta = \frac{N}{2} \frac{1}{\beta} \sum_{-\pi/a < \omega_n < \pi/a} \int_{-\pi/a}^{\pi/a} \frac{dp_1}{2\pi} \frac{\hat{\omega}_n^2 - \hat{p}_1^2}{\hat{\omega}_n^2 + \hat{p}_1^2 + \sigma_\beta}. \quad (\text{B2})$$

In this expression, we use the identity

$$\frac{1}{\beta} \sum_{-\pi/a < \omega_n < \pi/a} F(\omega_n) = \sum_{n=-\infty}^{\infty} \int_{-\pi/a}^{\pi/a} \frac{dp_0}{2\pi} e^{ip_0 \beta n} F(p_0) \quad (\text{B3})$$

to transform the sum over ω_n into the integral over p_0 . Then only the $n = 0$ term,

$$\frac{N}{2} \int_p \frac{\hat{p}_0^2 - \hat{p}_1^2}{\hat{p}^2 + \sigma_\beta}, \quad (\text{B4})$$

may potentially be UV divergent for $a \rightarrow 0$, but actually this term vanishes because of the hypercubic symmetry. Other $n \neq 0$ terms are UV convergent and we may remove the lattice regulator. In this

⁶ If one also applies the Noether method to the “measure term” (Eq. (2.16) of Ref. [12]), the energy–momentum tensor would have an additional term, $-(1/2)\delta^2(0)\delta_{\mu\nu} \ln[1 - \sum_{i=1}^{N-1} n^i(x)n^i(x)/N]$. This term, however, gives rise to only sub-leading contributions in the large- N limit and does not affect the following result.

way, we have

$$\left\langle T_{00}^{\text{naive}}(x) \right\rangle_{\beta} = -\left\langle T_{11}^{\text{naive}}(x) \right\rangle_{\beta} = N \sum_{n=1}^{\infty} \int \frac{d^2 p}{(2\pi)^2} e^{ip_0 \beta n} \frac{p_0^2 - p_1^2}{p^2 + \sigma_{\beta}} = -\frac{N}{2\pi} \sigma_{\beta} \sum_{n=1}^{\infty} K_2(\beta \sqrt{\sigma_{\beta}} n). \quad (\text{B5})$$

This reproduces the expectation value of the traceless part $\langle \{T_{00}\}_R(x) - \{T_{11}\}_R(x) \rangle_{\beta}$ correctly, but it misses the trace part $\langle \{T_{00}\}_R(x) + \{T_{11}\}_R(x) \rangle_{\beta} = -N/(4\pi)(\sigma_{\beta} - \sigma)$. This failure for the “trace anomaly” is expected, because the naive expression (B1) is traceless for $D = 2$ and it cannot reproduce the trace anomaly when lattice regularization in $D = 2$ is used. Our universal formula (5.2) can, on the other hand, incorporate the effect of the trace anomaly correctly, even with lattice regularization.

References

- [1] M. Lüscher, J. High Energy Phys. **1008**, 071 (2010) [[arXiv:1006.4518](#) [hep-lat]] [[Search INSPIRE](#)].
- [2] M. Lüscher, PoS LATTICE **2013**, 016 (2014) [[arXiv:1308.5598](#) [hep-lat]] [[Search INSPIRE](#)].
- [3] M. Lüscher and P. Weisz, J. High Energy Phys. **1102**, 051 (2011) [[arXiv:1101.0963](#) [hep-th]] [[Search INSPIRE](#)].
- [4] M. Lüscher, J. High Energy Phys. **1304**, 123 (2013) [[arXiv:1302.5246](#) [hep-lat]] [[Search INSPIRE](#)].
- [5] H. Suzuki, Prog. Theor. Exp. Phys. **2013**, 083B03 (2013) [[arXiv:1304.0533](#) [hep-lat]] [[Search INSPIRE](#)].
- [6] L. Del Debbio, A. Patella, and A. Rago, J. High Energy Phys. **1311**, 212 (2013) [[arXiv:1306.1173](#) [hep-th]] [[Search INSPIRE](#)].
- [7] H. Makino and H. Suzuki, Prog. Theor. Exp. Phys. **2014**, 063B02 (2014) [[arXiv:1403.4772](#) [hep-lat]] [[Search INSPIRE](#)].
- [8] S. Caracciolo, G. Curci, P. Menotti, and A. Pelissetto, Nucl. Phys. B **309**, 612 (1988).
- [9] S. Caracciolo, G. Curci, P. Menotti, and A. Pelissetto, Annals Phys. **197**, 119 (1990).
- [10] M. Asakawa et al. [FlowQCD Collaboration], Phys. Rev. D **90**, 011501 (2014) [[arXiv:1312.7492](#) [hep-lat]] [[Search INSPIRE](#)].
- [11] M. Kitazawa, M. Asakawa, T. Hatsuda, T. Iritani, E. Itou, and H. Suzuki, [[arXiv:1412.4508](#) [hep-lat]] [[Search INSPIRE](#)].
- [12] H. Makino and H. Suzuki, [[arXiv:1410.7538](#) [hep-lat]] [[Search INSPIRE](#)].
- [13] K. Kikuchi and T. Onogi, J. High Energy Phys. **1411**, 094 (2014) [[arXiv:1408.2185](#) [hep-th]] [[Search INSPIRE](#)].
- [14] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).
- [15] S. Aoki, K. Kikuchi, and T. Onogi, [[arXiv:1412.8249](#) [hep-th]] [[Search INSPIRE](#)].